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Canonical transformations of the extended phase space, Toda lattices and the Stäckel family of integrable systems

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Abstract. We consider some examples of canonical transformations of the extended phase space, which map a completely integrable system into another completely integrable system. The proposed transformations are closed to the known Maupertuis–Jacobi mappings and to the Kepler change of the time. Using the Kepler transformation we construct new integrable systems related to the Toda lattices and the Stäckel family of integrable systems.

1. Introduction

Let us begin with one simple example. Consider an ellipse defined by the standard implicit equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

One can represent this ellipse by the following parametric equations

$$x = a\sin(t) \qquad y = b\cos(t) \qquad t \in [0, 2\pi]. \tag{1.1}$$

It is known that there are infinitely many parametrizations of a given curve. For instance, we can reparametrize an ellipse by using another parameter

$$\tilde{t} = (a^2 + b^2)t + (a^2 - b^2)\sin(t). \tag{1.2}$$

Construction of the different polynomial, rational and other parametrizations of the plane curves is a subject of classical algebraic geometry.

In classical mechanics the same ellipse may be identified with integral trajectories of the various integrable systems on the common phase space. In this case parameter t is the time conjugated to some Hamilton function H. As an example, the first parametric form of the ellipse (1.1) is related to the two-dimensional oscillator, while the second parametrization (1.2) may be associated with the Kepler model [1].

Below we will consider integrable systems on the $2n$-dimensional symplectic manifold $\mathcal{M}$ with n integrals in the involution. According to the Liouville–Arnold theorem [1], integrability is a geometric property and, therefore, it does not depend on the choice of coordinates or parametrizations. So, starting with some known integrable system we can try to obtain new integrable models by using various parametric forms of the common trajectories. In this case we could expect that the initial and resulting integrable systems have many common properties.

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Two main problems are how to find a new parametric form of the initial trajectories and how to obtain a new Hamilton function defined on the whole phase space $\mathcal{M}$.

Finding a new parametrization of the known integral trajectories one has to introduce a new parameter $\tilde{t}$ and the corresponding Hamilton function $\tilde{H}$. Thus, we want to study the following mapping

$$t \mapsto \tilde{t} \qquad H(p,q) \mapsto \tilde{H}(p,q). \tag{1.3}$$

To consider such transformations let us extend initial phase space $\mathcal{M}$ with local coordinates $\{p_j, q_j\}_{j=1}^n$ by adding to it the new coordinate $q_{n+1} = t$ with the corresponding momenta $p_{n+1} = -H$. The resulting $(2n+2)$-dimensional space $\mathcal{M}_E$ [8, 16] is the so-called extended phase space of the Hamiltonian system. We underline that $H(p,q)$ is the Hamilton function on $\mathcal{M}$, but H is an independent variable in the space $\mathcal{M}_E$.

To describe evolution on the extended phase space $\mathcal{M}_E$ we introduce the generalized Hamilton function [8, 16]

$$\mathcal{H}(p_1, \ldots, p_{n+1}; q_1, \ldots, q_{n+1}) = H(p,q) - H. \tag{1.4}$$

The Hamilton equations for the variables $q_{n+1} = t$ and $p_{n+1} = -H$ look like

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = 1 \qquad \frac{\mathrm{d}H}{\mathrm{d}\tau} = 0.$$

Here τ is the generalized time (parameter) associated with generalized Hamilton function $\mathcal{H}$. So, the time variable t is a cyclic coordinate and the conjugated momentum is a constant of motion. The other $2n$ equations coincide with the initial Hamilton equations on the zero-valued energy surface

$$\mathcal{H}(p,q) = H(p,q) - H = 0. \tag{1.5}$$

Thus, our initial Hamiltonian system on $\mathcal{M}$ may be immersed into the Hamiltonian system on $\mathcal{M}_E$. Using this immersion and canonical transformations of the extended phase space $\mathcal{M}_E$ [8, 16] we obtain transformations

$$\begin{array}{ccc} (H,t) & \text{canonical transformations} & (\tilde{H},\tilde{t}) \\ \mathcal{M}_E & \longrightarrow & \mathcal{M}_E \\ \uparrow & & \downarrow \\ \mathcal{M} & & \mathcal{M} \\ \uparrow & & \downarrow \\ H(p,q) & & \tilde{H}(p,q) \end{array}$$

which map an initial Hamilton function $H(p,q) \mapsto \tilde{H}(p,q)$ into the other Hamilton function defined on the same phase space.

In this paper we consider some old and new examples of such transformations, which map an integrable system into the other integrable system. Note, two different classical definitions of the canonical transformations are known [1].

(1) *Canonical transformations preserve the canonical form of the Hamilton–Jacobi equations.*
(2) *Canonical transformations preserve the differential 2-form* $\Omega_2 = \sum_{i=1}^n \mathrm{d}p_i \wedge \mathrm{d}q_i$.

For instance, the first definition is used in textbooks on variational principles of classical mechanics [3, 6, 8, 16]. The second definition is included to consider the geometry of the phase space [1].

Below we will use the first definition of the canonical transformations because we do not know an appropriate description of the symplectic structure of the extended phase space $\mathcal{M}_E$ with imposed constraint (1.5). The second reason is that the known Maupertuis–Jacobi transformation and the Kepler change of the time preserve the form of the Hamilton–Jacobi equation, but they retain the corresponding differential 2-form $\Omega_2 = \sum_{i=1}^{n+1} \mathrm{d}p_i \wedge \mathrm{d}q_i$ on the constant energy level $H = E$ only [2].

Let us introduce general canonical transformations of the extended phase space $\mathcal{M}_E$

$$\begin{aligned} &t \mapsto \tilde{t} \qquad \mathrm{d}\tilde{t} = v(p,q)\,\mathrm{d}t \\ &H \mapsto \tilde{H} \qquad \tilde{H} = v(p,q)^{-1} H \end{aligned} \tag{1.6}$$

which change initial equations of motion

$$\frac{\mathrm{d}q_i}{\mathrm{d}\tilde{t}} = v^{-1}(p,q)\left(\frac{\mathrm{d}q_i}{\mathrm{d}t} - \tilde{H}\frac{\partial v}{\partial p_i}\right) \qquad \frac{\mathrm{d}p_i}{\mathrm{d}\tilde{t}} = v^{-1}(p,q)\left(\frac{\mathrm{d}p_i}{\mathrm{d}t} + \tilde{H}\frac{\partial v}{\partial q_i}\right)$$

but preserve the canonical form of the Hamilton–Jacobi equation

$$\frac{\partial \mathcal{S}}{\partial t} + H = 0 \qquad \text{where} \quad \mathcal{S} = \int p\,\mathrm{d}q - H\,\mathrm{d}t \tag{1.7}$$

and retain the corresponding zero-energy surfaces (1.5) at $v(p,q) \neq 0$

$$\tilde{\mathcal{H}}(p,q) = v(p,q)^{-1}\mathcal{H}(p,q) = 0.$$

Zeros of the function $v(p,q)$ determine the behaviour of the system with respect to the inversion of the time (see discussions in the textbooks on celestial [3] and relativistic mechanics [10]). Here we will not consider this problem in detail.

In the absence of the symplectic theory of $\mathcal{M}_E$ we have no regular method to obtain functions $v(p,q)$ defining canonical transformations of $\mathcal{M}_E$ (1.6), which preserve integrability. Up to now there is no rule for how to proceed: each case is different.

In the second section the known Maupertuis–Jacobi mapping and the Kepler change of the time are discussed. These transformations of the extended phase space $\mathcal{M}_E$ preserve integrability.

In the third section we will use the Kepler mapping to construct canonical transformations of $\mathcal{M}_E$, which map Toda lattices into the new integrable systems.

In the fourth section we consider another generalization of the Kepler canonical transformations of $\mathcal{M}_E$. For instance, we discuss integrable systems with the following Hamilton functions:

$$\begin{aligned} &H = p_x^k p_y^k + \alpha(xy)^{-\frac{k}{k+1}} \qquad \alpha, k \in \mathbb{R} \\ &H = p_x^k + p_y^k + \alpha(xy)^{-\frac{k}{k+1}} \end{aligned} \tag{1.8}$$

where α and k are arbitrary parameters. At $k = 1$ the first Hamiltonian coincides with the Hamiltonian of the Kepler problem in rotated variables $q_{1,2} = x \pm y$. At $k = 2$ the second integrable Hamiltonian has been found by Fokas and Lagerstrom [5].

2. The Maupertuis–Jacobi mapping and the Kepler change of the time

Let us consider the natural mechanical system determined by the Hamilton function

$$H(p,q) = \sum_{i,j}^{n} g_{ij}(q) p_i p_j + V(q). \tag{2.1}$$

Due to the Maupertuis principle [1,2], on the fixed $(2n-1)$-dimensional smooth manifold

$$Q^{2n-1} = (H(p,q) = E,\ E > \max V(x)) \tag{2.2}$$

trajectories of the vector field $X = \text{sgrad}\, H(p,q)$ coincide with trajectories of another vector field $\tilde{X} = \text{sgrad}\, \tilde{H}(p,q)$, where the new Hamilton function is given by

$$\tilde{H}(p,q) = \sum_{i,j}^{n} \frac{g_{ij}(q)}{E - V(q)} p_i p_j. \tag{2.3}$$

The so-called Maupertuis transformation

$$X \mapsto \tilde{X}$$

maps initial Hamiltonian vector field X on $\mathcal{M}$ into the other Hamiltonian vector field $\tilde{X}$ defined on the same manifold $\mathcal{M}$ [2].

Let t be the time along trajectories of the initial vector field X and $\tilde{t}$ be the time along trajectories of the new vector field $\tilde{X}$. The Maupertuis mapping [2] gives rise to so-called Jacobi transformations [8, 13]

$$t \mapsto \tilde{t} \qquad H(p,q) \mapsto \tilde{H}(p,q)$$

such that

$$\mathrm{d}\tilde{t} = (E - V(q))\,\mathrm{d}t. \tag{2.4}$$

The main useful property of the Maupertuis–Jacobi transformations is that any integrable system with a natural Hamiltonian $H(p,q)$ may be mapped into other integrable systems on the same phase space. The initial and the resulting integrable systems are topologically equivalent [2]. This property has been used for the search for new integrable systems (see references within [2, 6, 13, 14]). The same property for the integrable system with non-natural Hamilton function is discussed in [11].

On the surface Q^{2n-1} (2.2) differentials of the initial and resulting Hamilton functions satisfy the following equation:

$$\mathrm{d}\tilde{H}(p,q) = (E - V(q))^{-1}\,\mathrm{d}H(p,q).$$

Therefore, the Maupertuis–Jacobi transformations preserve the corresponding differential 2-form $\Omega_2 = \sum_{i=1}^{n+1} \mathrm{d}p_i \wedge \mathrm{d}q_i$ almost everywhere [2].

The Maupertuis–Jacobi mapping (2.3), (2.4) may be rewritten as the canonical transformation (1.6) of the extended phase space

$$T(p,q) \mapsto \tilde{T}(p,q) = v(p,q)^{-1} T(p,q) \qquad v(p,q) = E - V(q)$$

which maps the initial geodesic flow into another geodesic flow. This map preserves integrability, if the function $v(p,q)$ is constructed by any potential $V(q)$, which may be added to the initial kinetic energy $T(p,q)$ without loss of integrability [6].

In contrast with the Maupertuis–Jacobi transformations, even if the general canonical transformation of $\mathcal{M}_E$ (1.6) preserves integrability, we do not have a general method to construct new integrals of motion starting with initial ones. However, we could use some known examples of such transformations [2, 7, 14, 19–21].

Let us begin with the two-dimensional oscillator defined by the Hamilton function

$$H_{\text{osc}}(p,q) = p_1^2 + p_2^2 + a(q_1^2 + q_2^2) + b.$$

For this system the Kepler canonical transformation (1.6) of $\mathcal{M}_E$ with the function

$$v(p,q) = q_1^2 + q_2^2$$

preserves integrability. After change of the time (1.2), (1.6) and the point canonical transformation of the other variables

$$x = q_1 q_2 \qquad y = (q_1^2 - q_2^2)/2 \tag{2.5}$$

the orbits of the oscillator map into the orbits of the Kepler problem

$$\tilde{H}_{\text{kepl}}(p,x) = \frac{H_{\text{osc}}(p,q)}{q_1^2+q_2^2} = p_x^2 + p_y^2 + \frac{b}{2\sqrt{x^2+y^2}} + a. \tag{2.6}$$

Namely this Kepler change of the time is used to integrate equations of motion. The Kepler change of the time has been generalized by Liouville [9].

The oscillator and the Kepler model belong to the Stäckel family of integrable systems. According to [19], we can construct a generalization of the Kepler canonical transformations (1.6) of the extended phase space $\mathcal{M}_E$, which map any integrable Stäckel system into another integrable Stäckel system.

For the uniform Stäckel systems, if we bring equations of motion into the Lax form

$$\{H(p,q), L(\lambda)\} = [L(\lambda), A(\lambda)]$$

the generalized Kepler transformations [19] give rise to the following mappings of the Lax matrices:

$$L(\lambda) \mapsto \tilde{L}(\lambda) = L(\lambda) + \tilde{H}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \tilde{A}(\lambda) = v(p,q)^{-1}A(\lambda). \tag{2.7}$$

Transformations of the corresponding spectral curves $\mathcal{C} : \det(L(\lambda) - \mu I) = 0$ look like

$$\begin{array}{ll} \mathcal{C}: & \mu^2 = \sum a_j\lambda^j + a_m\lambda^m + H\lambda^k + \sum J_i\lambda^i \\ \tilde{\mathcal{C}}: & \mu^2 = \sum a_j\lambda^j + \tilde{H}\lambda^m + a_k\lambda^k + \sum \tilde{J}_i\lambda^i. \end{array} \tag{2.8}$$

Here n coefficients $H(p,q)$, $J_i(p,q)$ and $\tilde{H}(p,q)$, $\tilde{J}_i(p,q)$ are integrals of motion. Other coefficients $a_j \in \mathbb{R}$ are arbitrary parameters (charges), which define the potential part of the Hamilton function $H(p,q) = T(p) + V(q,a)$. Canonical transformations of $\mathcal{M}_E$ give rise to permutation of the Hamilton function $H(p,q)$ and one of the parameters $a_k \in \mathbb{R}$ (charge). Other characteristics of the algebraic curves are invariant with respect to transformations (1.6).

We can obtain trajectories of the initial and resulting systems by using the Abel–Jacobi method and the first kind of Abelian differential on the spectral curves $\mathcal{C}$ and $\tilde{\mathcal{C}}$, respectively [19]. The Riemann surfaces $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are topologically equivalent and, therefore, integrable systems related by the Kepler canonical transformations are topologically equivalent. Moreover, at the special choice of the values of integrals and parameters a_i the initial curve $\mathcal{C}$ will be equal to the resulting curve $\tilde{\mathcal{C}}$. In this case trajectories of the initial system coincide with trajectories of the resulting system. Recall that the Maupertuis–Jacobi mapping has the same property [2].

In [20] we used similar mappings of the Lax matrices and of the spectral curves to study the Holt systems, one of which does not belong to the Stäckel family of integrable systems. In the next section we will apply similar mappings to the Toda lattices.

3. The Toda lattices

Before proceeding further, it is useful to recall some known facts about generalized Toda lattices (all details may be found in the review [12]).

Let $\mathfrak{g}$ be a real, split, simple Lie algebra rank $\mathfrak{g} = n$. Let $K(,)$ be its Killing form, let $\mathfrak{a}$ be a split Cartan subalgebra, Δ the associated root system, Δ_+ the set of positive roots and P the system of simple roots. For $\alpha \in \Delta_+$, let $\mathfrak{g}_\alpha$ be the corresponding root space and $e_\alpha \in \mathfrak{g}_\alpha$ a root vector. It will be convenient to normalize e_α in such a way that $K(e_\alpha, e_{-\alpha}) = 1$.

The root space decomposition, $\mathfrak{g} = \mathfrak{a} + \oplus_\alpha \mathfrak{g}_\alpha$, gives rise to a natural grading on $\mathfrak{g}$, the so-called principal grading. Let $\mathfrak{g}_+ = \oplus_{i\geqslant 0}\mathfrak{g}_i$ be a Borel subalgebra of $\mathfrak{g}$ and $\mathfrak{g}_-$ be the opposite nilpotent subalgebra. The Killing form K allows us to identify $\mathfrak{g}_i^*$ with $\mathfrak{g}_{-i}$, so that $\mathfrak{g}_+^* = \oplus_{i\leqslant 0}\mathfrak{g}_i$ and $\mathfrak{g}_-^* = \oplus_{i>0}\mathfrak{g}_i$. Choose a vector

$$a = \sum_{\alpha\in P} a_a e_{-\alpha} \qquad a_\alpha \in \mathbb{R}$$

and let $\mathcal{O}_a$ be the $\mathfrak{g}_+$ orbit of a in $\mathfrak{a} + \mathfrak{g}_{-1}$. The points of $\mathcal{O}_a$ have the form

$$\xi = p + \sum_{\alpha\in P} c_\alpha a_\alpha e_{-\alpha} \qquad p \in \mathfrak{a} \qquad c_\alpha > 0.$$

The orbit $\mathcal{O}_a$ is parametrized by the canonical variables $\{p_j, q_j\}$ as follows:

$$\xi = \sum_{i=1}^n p_i h_i + \sum_{\alpha\in P} a_\alpha \cdot \exp\left(\sum_{i=1}^n q_i K(\alpha, h_i')\right) \cdot e_{-\alpha}.$$

For the Toda lattice the orbit $\mathcal{O}_a$ is an orbit of $\mathfrak{g}_+$; what we are really interested in is the orbits of the full algebra $\mathfrak{g}_R = \mathfrak{g}_+ \oplus \mathfrak{g}_-$. Let us translate $\mathcal{O}_a$ by adding to it a constant vector $e = \sum_{\alpha\in P} e_\alpha$ which may be regarded as a one-point orbit of $\mathfrak{g}_-$. The resulting orbit

$$\mathcal{O}_{ae} = \mathcal{O}_a + e \tag{3.1}$$

is parametrized by the canonical variables as follows:

$$L = \sum_{i=1}^n p_i h_i + \sum_{\alpha\in P} a_\alpha \cdot \exp K(\alpha, q) \cdot e_{-\alpha} + \sum_{\alpha\in P} e_\alpha. \tag{3.2}$$

Let us consider the simplest Hamiltonian on $\mathcal{O}_{ae}$ generated by the Killing form on $\mathfrak{g}$

$$H(X) = \tfrac{1}{2}K(X, X). \tag{3.3}$$

Its restriction to $\mathcal{O}_{ae}$ (3.2) is given by

$$H(p, q) = \tfrac{1}{2}K(p, p) + \sum_{\alpha\in P} a_\alpha \mathrm{e}^{\alpha(q)}. \tag{3.4}$$

To keep in mind the consequent canonical transformation of $\mathcal{M}_E$, we choose the second Lax matrix at the following special form:

$$\frac{\mathrm{d}}{\mathrm{d}t} L = [L, A] \qquad A = -\sum_{\alpha\in P} a_\alpha \cdot \exp K(\alpha, q) \cdot e_{-\alpha}. \tag{3.5}$$

Now let us construct canonical transformations of the extended phase space $\mathcal{M}_E$ for these Toda lattices. Recall, in a shifted version of the Adler–Kostant–Symes scheme in order to obtain orbit $\mathcal{O}_{ae}$ (3.1) in $\mathcal{M} \simeq \mathfrak{g}_R^* = \mathfrak{g}_+^* \oplus \mathfrak{g}_-^*$ we translate orbit $\mathcal{O}_a$ living in $\mathfrak{g}_+^*$ by adding to it a constant vector e from the remaining part of $\mathcal{M}$. Let us replace the phase space $\mathcal{M}$ on the extended phase space $\mathcal{M}_E$. In this case we can also translate the same orbit $\mathcal{O}_a$ in $\mathfrak{g}_+^*$ by adding to it a constant vector from the remaining part of the whole space $\mathcal{M}_E$. As above, this vector has to be a character and a constant with respect to the new time.

The third condition is that the initial invariant polynomial (3.3) has to generate the coupling constant

$$K(\tilde{L}, \tilde{L}) = -b \tag{3.6}$$

instead of the Hamiltonian $\tilde{H}$, as for the Stäckel systems (2.8) [19]. Namely, this condition together with the form of transformations of the Lax matrices (2.7) dictates a very special choice of the functions $v(p, q)$ in (1.6) for the Toda lattices.

Proposition 1. *For each simple root $\beta \in P$ and for any constant $b_\beta \in \mathbb{R}$ the following canonical transformation of the extended phase space $\mathcal{M}_E$*

$$\begin{aligned} \mathrm{d}\tilde{t} &= \mathrm{e}^{\beta(q)} \cdot \mathrm{d}t \\ \tilde{H}_\beta &= \mathrm{e}^{-\beta(q)} \cdot (H + b_\beta) \end{aligned} \tag{3.7}$$

maps the Toda lattice into another integrable system. This canonical transformation induces the following transformation of the Lax matrices:

$$\tilde{L}_\beta = L - \tilde{H}_\beta \cdot \frac{e_\beta}{a_\beta} \qquad \tilde{A} = \mathrm{e}^{-\beta(q)} \cdot A \tag{3.8}$$

such that

$$\{\tilde{H}, \tilde{L}\} = [\tilde{L}, \tilde{A}] \qquad \textit{and} \qquad K(\tilde{L}, \tilde{L}) = -b_\beta. \tag{3.9}$$

Here H, L and L_- are the Hamiltonian (3.4) and the Lax matrices (3.2), (3.5) for the corresponding Toda lattice.

The new Lax equation (3.9) is directly verified

$$\begin{aligned} \{\tilde{H}, \tilde{L}\} = \{\tilde{H}, L\} &= \mathrm{e}^{-\beta(q)}\{H, L\} + (H + b_\beta)\left\{\mathrm{e}^{-\beta(q)}, \sum_{j=1}^{n} p_j h_j\right\} \\ &= [L, \mathrm{e}^{-\beta(q)} A] - \tilde{H} \cdot \left[\frac{e_\beta}{a_\beta}, \mathrm{e}^{-\beta(q)} A\right] = [\tilde{L}, \tilde{A}]. \end{aligned}$$

There are $n = \operatorname{rank} \mathfrak{g}$ functionally independent invariant polynomials on $\mathfrak{g}$. Restricted to the orbit $\mathcal{O}_{ae}$ they remain functionally independent and give rise to the integral of motion for the Toda lattice. In (3.8) we translate $\mathcal{O}_{ae}$ by adding to it a 'constant' vector proportional to the element of the universal enveloping algebra. All the invariant polynomials are invariant with respect to this transformation. Thus, we can construct n independent integrals of motion in the involution for the system with the Hamilton function (3.7).

To construct the new Lax matrix (3.8) we used some inner analogies with the Kepler problem (3.6), (2.7). Let us consider some heuristic algebraic arguments concerning the Lax matrix transformation (3.8). Recall, constructions of the orbit $\mathcal{O}_{ae}$ and the corresponding classical R-matrix are closely related to the principal grading of $\mathfrak{g}$. This grading defines all the possible embeddings of the three-dimensional subalgebra $A_1 \simeq sl(2)$ into $\mathfrak{g}$.

Let $\{\boldsymbol{e}_-, \boldsymbol{e}_+, \boldsymbol{h}\}$ be generators of the Lie algebra $sl(2)$

$$[\boldsymbol{h}, \boldsymbol{e}_-] = \boldsymbol{e}_- \qquad [\boldsymbol{h}, \boldsymbol{e}_+] = -\boldsymbol{e}_+ \qquad [\boldsymbol{e}_-, \boldsymbol{e}_+] = 2\boldsymbol{h} \tag{3.10}$$

and the element

$$\boldsymbol{\Delta} = \boldsymbol{h}^2 + \tfrac{1}{2}(\boldsymbol{e}_-\boldsymbol{e}_+ + \boldsymbol{e}_+\boldsymbol{e}_-) \tag{3.11}$$

of the universal enveloping algebra be the Laplace operator in $SL(2)$. Let us consider the infinite-dimensional irreducible representation $\mathcal{W}$ of the Lie algebra $sl(2)$ in the linear space V such that

$$\mathcal{W} : \{\boldsymbol{e}_-, \boldsymbol{e}_+, \boldsymbol{h}\} \to \{e, f, h\} \in \operatorname{End}(V).$$

Let operator e be invertible in End (V) and $\varphi(\Delta)$ be an arbitrary function on the value of the Casimir operator (3.11); then the mapping

$$e_- \to e'_- = e_- \qquad h \to h' = h \qquad e_+ \to e'_+ = e_+ + e_-^{-1} \cdot \varphi(\Delta) \tag{3.12}$$

is an outer automorphism of the space of infinite-dimensional representations of $sl(2)$ in V [18]. These mappings shift the spectrum of the Laplace operator $\boldsymbol{\Delta}$ (3.11) by the rule

$$\Delta \to \Delta' = \Delta + \varphi(\Delta).$$

In particular by $\varphi(\Delta) = -(\Delta + b)$ one obtains $\Delta \to \Delta' = -b$. So, instead of the spectrum of the Laplace operator $\mathbf{\Delta}$ on the group $SL(2)$ one obtains the spectrum of the coupling constant b.

Now we turn to the Toda lattices related to the Lie algebra $\mathfrak{g}$, which contains subalgebras A_1 associated with the roots $\beta \in P$. The maps (3.9) may be closely related to the automorphism (3.12). Instead of the restriction Δ (3.11) of the Casimir operator $H(X)$ (3.3) on A_1 one has to substitute restriction $H(p, q)$ (3.4) of the same Casimir operator on the orbit $\mathcal{O}_{a,e}$. Note that the outer automorphism (3.12) non-trivially acts on the one nilpotent subalgebra of $sl(2)$ only.

Of course, the more justified consideration of the canonical transformation of the extended phase space $\mathcal{M}_E$ requires a more advanced technique of representation theory.

The number of functionally independent Hamilton functions $\tilde{H}_\beta$, $\beta \in P$ depends on the symmetries of the associated root system. For the closed Toda lattices associated with the affine Lie algebras the canonical time transformation has a similar form. The spectral curves associated with the Lax matrices L (3.2) and $\tilde{L}$ (3.8) depend on the choice of representation of $\mathfrak{g}$. Therefore, geometric transformations of the spectral curves may be considered on some examples only [21].

In conclusion, we present integrable systems related to the two-particle Toda lattices associated with affine algebras $X_2^{(1)}$. After an appropriate point transformation of coordinates (similar to (2.5), see [21]), all the Hamilton functions have a common form

$$\tilde{H} = p_x p_y + \frac{b}{xy} + a x^{z_1} y^{z_2} + c x^{s_1} y^{s_2} + d \qquad a, b, c, d \in \mathbb{R} \tag{3.13}$$

where $z_{1,2}$ and $s_{1,2}$ are the roots of the different quadratic equations, which may be related to the angles of the corresponding Dynkin diagrams. Below we show these equations only:

$$\begin{array}{lll}
A_3^{(1)}: & z^2 + 3z + 3 = 0 & s^2 + 3s + 3 = 0 \\[2ex]
B_2^{(1)}\ C_2^{(1)}: & z^2 + 4z + 5 = 0 & s^2 + 4s + 5 = 0 \\
 & z^2 + 2z + 2 = 0 & s^2 + 3s + \frac{5}{2} = 0 \\[2ex]
D_2^{(1)}: & z^2 + 2z + 2 = 0 & s^2 + 2s + 2 = 0 \\
 & z^2 + 2z + 2 = 0 & (s + 2)^2 = 0 \\[2ex]
 & z^2 + 2z + 4 = 0 & s^2 + 5s + 7 = 0 \\
G_2^{(1)}: & z^2 + 2z + 4 = 0 & s^2 + 3s + 3 = 0 \\
 & z^2 + 3z + \frac{7}{3} = 0 & s^2 + 3s + 3 = 0.
\end{array}$$

An integrable system with the Hamilton function $\tilde{H}$ (3.13) associated with the root system $A_3^{(1)}$ was found for the first time in [4].

The corresponding second integrals of motion K are polynomials third, fourth and sixth order in momenta. Note, for the algebra $A_3^{(1)}$ all three Hamiltonians H_β, $\beta \in P$ are equivalent. Two different Hamilton functions (3.7), (3.13) are associated with the algebras $B_2^{(1)}$, $C_2^{(1)}$ and $D_2^{(1)}$. For the $G_2^{(1)}$ algebra we have three different Hamiltonians (3.7), (3.13).

Some properties of the proposed canonical transformations of $\mathcal{M}_E$ for the Toda lattices are considered in [21].

4. The Stäckel systems

Let us return to the Kepler transformation (2.6) of $\mathcal{M}_E$. After permutation coordinates and momenta ($q_{1,2} \leftrightarrow p_{1,2}$) the Hamilton function for the Kepler problem

$$\tilde{H} = a\frac{p_1^2 + p_2^2 + a^{-1}(q_1^2 + q_2^2) + a^{-1}b}{p_1^2 + p_2^2} = a\frac{H_{\mathrm{osc}}}{H_{\mathrm{free}}}$$

becomes a ratio of the Hamilton functions H_{osc} for the oscillator and H_{free} for the free motion. So, for the two integrable systems the ratio of their Hamilton functions could be the Hamilton function of the third integrable system on the same phase space.

In this section we propose an interesting extension of the integrable family of the Stäckel systems [15] by using this observation (see also the review [7] and references therein).

Let us consider two integrable Hamiltonian systems on the common phase space $\mathcal{M}$. These systems are defined by the two sets of independent integrals of motion $\{I_j\}_{j=1}^n$ and $\{J_j\}_{j=1}^n$, in the involution

$$\{I_j, I_k\} = 0 \qquad \text{and} \qquad \{J_j, J_k\} = 0 \qquad j, k = 1, \ldots, n.$$

By using the inner product of the two independent vectors of integrals $\boldsymbol{I}$ and $\boldsymbol{J}$ in $\mathbb{R}^n$ we introduce the antisymmetric matrix $\mathcal{K} = (\boldsymbol{I} \wedge \boldsymbol{J})$. Any column or row of this matrix defines a set of the $n-1$ independent functions

$$\mathcal{K}_{ij} = (\boldsymbol{I} \otimes \boldsymbol{J})_{ij} = I_i J_j - I_j J_i \qquad i, j = 1, \ldots, n.$$

Proposition 2. *If all the differences of integrals of motion $(I_j - J_j)$ with the common index $j = 1, \ldots, n$ are in the involution*

$$\{I_j - J_j, I_k - J_k\} = 0 \qquad j, k = 1, \ldots, n \tag{4.1}$$

then the ratio of integrals

$$K_m = \frac{I_m}{J_m} \tag{4.2}$$

and $n-1$ functions K_j,

$$K_j = \frac{\mathcal{K}_{mj}}{J_m} = \frac{I_m J_j - I_j J_m}{J_m} = K_m J_j - I_j \qquad m \neq j = 1, \ldots, n \tag{4.3}$$

are integrals of motion for new integrable system on the same phase space.

By definition all the new integrals K_m and K_j are functionally independent. Thus, the proof is a straightforward calculation of the following Poisson brackets:

$$\{K_m, K_j\} = \frac{I_m}{J_m^2}(\{I_m, J_j\} + \{J_m, I_j\}) = -\frac{I_m}{J_m^2}\{I_m - J_m, I_j - J_j\} = 0$$

and

$$\begin{aligned}\{K_j, K_k\} &= K_m\{J_j, K_k\} - \{I_j, K_k\}\\ &= J_k(K_m\{J_j, K_m\} - \{I_j, K_m\}) - K_m(\{J_j, I_k\} + \{I_j, J_k\})\\ &= J_k(\{K_j, K_m\} + K_m\{I_j - J_j, I_k - J_k\} = 0 \qquad j \neq k \neq m.\end{aligned}$$

Thus, mappings (4.2), (4.3) define the canonical transformation (1.6) of the extended phase space, which preserves the property of integrability. To apply this transformation we have to find two integrable systems satisfying condition (4.1). Below we prove that the Stäckel integrable systems [15] may be considered as the main example of the systems satisfying condition (4.1).

Let us briefly recall some necessary facts about the Stäckel systems [15, 17]. The non-degenerate $n \times n$ Stäckel matrix $\boldsymbol{S}$, whose j column s_{kj} depends only on q_j

$$\det \boldsymbol{S} \neq 0 \qquad \frac{\partial s_{kj}}{\partial q_m} = 0 \qquad j \neq m$$

defines functionally independent integrals of motion $\{I_k\}_{k=1}^n$

$$I_k = \sum_{j=1}^n c_{jk}(p_j^2 + U_j(q_j)) \qquad c_{jk} = \frac{\boldsymbol{S}_{kj}}{\det \boldsymbol{S}} \tag{4.4}$$

which are quadratic in momenta. Here $\boldsymbol{C} = [c_{jk}]$ denotes the inverse matrix to $\boldsymbol{S}$ and $\boldsymbol{S}_{kj}$ is a cofactor of the element s_{kj}.

According to [19], if the two Stäckel matrices $\boldsymbol{S}$ and $\tilde{\boldsymbol{S}}$ are distinguished by the mth row only

$$s_{kj} = \tilde{s}_{kj} \qquad k \neq m$$

the corresponding Stäckel systems with a common set of potentials U_j and with the Hamilton functions I_m and $\tilde{I}_m$ are related by the canonical change of the time

$$\tilde{I}_m \longleftrightarrow I_m \qquad \tilde{I}_m = \frac{I_m(p, q)}{v(q)} \tag{4.5}$$

where

$$v(q_1, \ldots, q_n) = \frac{\det \tilde{\boldsymbol{S}}(q_1, \ldots, q_n)}{\det \boldsymbol{S}(q_1, \ldots, q_n)}.$$

The canonical transformation (4.5) connects two Stäckel systems with different Stäckel matrices and with the common set of potentials U_j.

Let us consider a pair of the Stäckel systems with a common Stäckel matrix $\boldsymbol{S}$ and with different potentials. Namely, in addition to the system with integrals $\{I_k\}$ (4.4), we introduce the second integrable system with the similar integrals of motion

$$J_k = \sum_{j=1}^n c_{jk}(p_j^2 + W_j(q_j)) \qquad k = 1, \ldots, n. \tag{4.6}$$

Here even one potential $U_j(q_j)$ has to be functionally independent of the corresponding potential $W_j(q_j)$.

Proposition 3. *Any two integrable systems defined by the same Stäckel matrix $\boldsymbol{S}$ and by the functionally independent potentials $U_j(q_j)$ and $W_j(q_j)$ satisfy the necessary condition (4.1) of the previous proposition. Thus, the ratio of the two Stäckel integrable Hamiltonians defines the new integrable system*

$$K_m \longleftrightarrow (I_m, J_m) \qquad K_m = \frac{I_m}{J_m}. \tag{4.7}$$

It is obvious that all the integrals I_k and J_k differ by the potential part

$$(I_k - J_k) = \sum_{j=1}^n c_{jk}[U_j(q_j) - W_j(q_j)]$$

depending on coordinates $\{q_j\}$ only. Thus, systems with a common Stäckel matrix $\boldsymbol{S}$ satisfy condition (4.1).

The Hamilton function (4.7) has the following form:

$$H = K_m = \frac{\sum_{j=1}^n c_{jm}[p_j^2 + U_j(q_j)]}{\sum_{j=1}^n c_{jm}[p_j^2 + W_j(q_j)]}. \tag{4.8}$$

This Hamiltonian H is a rational function in momenta, but next one can try to use canonical transformations to simplify it. In a rare case, one obtains again a natural type Hamilton function as will be shown below.

As for the usual Stäckel system, the common level surface of the new integrals (4.2)

$$M_\alpha = \{z \in \mathbb{R}^{2n} : K_j(z) = \alpha_j,\ j = 1, \dots, n\}$$

is diffeomorphic to the real torus. Namely, substituting

$$V_j(q_j) = (1 - \alpha_m)^{-1}(U_j(q_j) - \alpha_m W_j(q_j))$$

into the definitions (4.4), (4.6) and (4.8) we obtain the following equations:

$$\begin{aligned} &\sum_{j=1}^{2} c_{jm}[p_j^2 + V_j(q_j)] = 0 = \beta_m \\ &\sum_{j=1}^{2} c_{jk}[p_j^2 + V_j(q_j)] = -\frac{\alpha_k}{1-\alpha_m} = \beta_k. \end{aligned} \tag{4.9}$$

After multiplication of these equations by the Stäckel matrix one immediately obtains

$$p_j^2 = \left(\frac{\partial \mathcal{S}_0}{\partial q_j}\right)^2 = \sum_{k=1}^{n} \beta_k s_{kj}(q_j) - V_j(q_j)$$

where $\mathcal{S}_0(q_1, \dots, q_n)$ is a reduced action function. The corresponding Hamilton–Jacobi equation on M_α

$$\frac{\partial \mathcal{S}_0}{\partial t} + H\left(t, \frac{\partial \mathcal{S}_0}{\partial q_1}, \dots, \frac{\partial \mathcal{S}_0}{\partial q_n}, q_1, \dots, q_n\right) = 0 \quad \Rightarrow \quad c_{jm}\frac{\partial \mathcal{S}_0}{\partial q_j}\frac{\partial \mathcal{S}_0}{\partial q_j} = E$$

admits the variable separation $\mathcal{S}_0(q_1, \dots, q_n) = \sum_{j=1}^{n} \mathcal{S}_j(q_j)$, where

$$\mathcal{S}_j(q_j) = \int \sqrt{\sum_{k=1}^{n} \beta_k s_{kj}(q_j) - V_j(q_j)}\, \mathrm{d}q_j.$$

Thus, coordinates $q_j\,(t, \alpha_1, \dots, \alpha_n)$ are determined from the equations

$$\sum_{j=1}^{n} \int \frac{s_{kj}(\lambda)\, \mathrm{d}\lambda}{\sqrt{\sum_{k=1}^{n} \beta_k s_{kj}(\lambda) - V_j(\lambda)}} = \delta_k \qquad k = 1, \dots, n$$

where the constants α_j and β_j are related by (4.9).

Thus, the solution of the problem is reduced to solving a sequence of one-dimensional problems, which is the essence of the method of separation of variables. Next, the integration problem for equation of motion is reduced to solution of the inverse Jacobi problem in the framework of algebraic geometry [19]. The corresponding algebraic curves are topologically equivalent, so the initial system is topologically equivalent to the resulting system [2].

4.1. Examples

As we have mentioned before, the Hamiltonian H (4.8) has a rather unusual expression. However, in some cases suitable canonical transformations can reduce it to a sum of the kinetic energy and the potential energy. Note that the necessary choice of such transformations is a generic problem for all the Stäckel systems. Thus, in this section we present several concrete two-dimensional systems only.

Let us consider a polar coordinate system on a plane with the usual coordinates (p_r, r) and (p_ϕ, ϕ) instead of $(p_{1,2}, q_{1,2})$, respectively. We take the first system with the axially symmetric potential

$$I_1 = p_r^2 + \frac{p_\phi^2}{r^2} - a^2 r^{2k} + b \qquad I_2 = p_\phi \qquad a, b, n \in \mathbb{R}. \tag{4.10}$$

The second system is associated with a free motion

$$J_1 = p_r^2 + \frac{p_\phi^2}{r^2} \qquad J_2 = p_\phi. \tag{4.11}$$

These systems belong to the Stäckel family of integrable systems with the following common Stäckel matrix:

$$\boldsymbol{S} = \begin{pmatrix} 1 & 0 \\ -r^{-2} & 1 \end{pmatrix}.$$

In this case $s_{12} = c_{21} = 0$ and $I_2 - J_2 = 0$; this allows us to use the second integrals in the non-Stäckel form (4.4).

The ratio (4.2) of the Hamiltonians (4.10) and (4.11) may be rewritten in the form (1.8) by using the set of canonical transformations. Let us begin with the usual transformation of the curvilinear coordinates to the Cartesian coordinates

$$\begin{aligned} &r = \sqrt{uv} \qquad \phi = \mathrm{i} \arctan\left(\frac{u - v}{u + v}\right) \\ &p_r = -\frac{up_u + vp_v}{r} \qquad p_\phi = \mathrm{i}(up_u - vp_v). \end{aligned} \tag{4.12}$$

Then, one permutes coordinates and momenta $(u \leftrightarrow p_u)$ and $(v \leftrightarrow p_v)$ such that the new Hamiltonian (4.8) becomes polynomial in momenta

$$H = \frac{I_1}{J_1} = -\frac{a^2}{4}\frac{p_u^k p_v^k}{uv} + \frac{b}{4uv} + 1. \tag{4.13}$$

In conclusion, we have to use the point canonical transformation

$$\begin{aligned} &p_x = p_u u^{\frac{1}{k+1}} \qquad x = (1 + 1/k)\, u^{\frac{k}{k+1}} \\ &p_y = p_v v^{\frac{1}{k+1}} \qquad y = (1 + 1/k)\, v^{\frac{k}{k+1}} \end{aligned}$$

which converts the Hamiltonian (4.13) into the following form:

$$H = p_x^k p_y^k + \alpha(xy)^{-\frac{k}{k+1}} + \beta$$

after multiplication on a suitable constant and rescaling parameters.

Note that the Stäckel matrix $\boldsymbol{S}$ and the set of potentials $U_j(q_j)$ are determined on the half of the phase space $\mathbb{R}^{2n}$ and depend on coordinates q_j only. Thus, we have some freedom related to the different canonical transformations of momenta

$$\begin{aligned} &(p_1, \ldots, p_n) \to (\tilde{p}_1, \ldots, \tilde{p}_n) \\ &p_i - \tilde{p}_i = 2\frac{\partial F(q_1, \ldots, q_n)}{\partial q_i} \qquad p_i + \tilde{p}_i = 0. \end{aligned} \tag{4.14}$$

Here $F(q_1, \ldots q_n)$ is a generating function of the transformations (4.14) depending on coordinates $\{q_j\}$, which are invariant with respect to transformation (4.14).

If condition (4.1) is invariant under these canonical transformations, we can apply them (4.14) to construct new integrable systems by the rule (4.2). Although no general procedure exists for this, one interesting case is known.

As above, one takes a system with the axially symmetric potential (4.10) and a system associated with a free motion with integrals

$$J_1 = \tilde{p}_r^2 + \frac{\tilde{p}_\phi^2}{r^2} \qquad J_2 = \tilde{p}_\phi. \tag{4.15}$$

New momenta $(\tilde{p}_r, \tilde{p}_\phi)$ relate to old ones (p_r, p_ϕ) by canonical transformation (4.14)

$$\tilde{p}_r = p_r - \frac{\partial f(r)}{\partial r}\frac{\cos(n\phi)}{n} \qquad \tilde{p}_\phi = p_\phi + f(r)\sin(n\phi). \tag{4.16}$$

Here $f(r)$ is any function on variable r and n are arbitrary parameters.

Both these systems belong to the Stäckel family of integrable system associated with a common Stäckel matrix. In this case $I_2 - J_2 \neq 0$ and condition (4.1) are not invariant by transformation (4.16). Let the second integrals be the square roots from the usual Stäckel integrals (4.4). If this form of the second integrals is used, the pair of systems (4.10) and (4.15) satisfies condition (4.1) by

$$\frac{f'(r)}{f(r)} = \frac{n}{r} \;\Rightarrow\; f(r) = cr^n \qquad c \in \mathbb{R}.$$

Let us move over to the Cartesian coordinates (4.12) and conjugated momenta

$$\tilde{p}_r = -\frac{up_u + vp_v}{r} \qquad \tilde{p}_\phi = \mathrm{i}(up_u - vp_v). \tag{4.17}$$

After permutation of coordinates and momenta ($u \leftrightarrow p_u$) and ($v \leftrightarrow p_v$) (the analogue of Fourier transformation in quantum mechanics), one obtains

$$H = \frac{I_1}{J_1} = \frac{1}{4uv}(c^2 p_u^{n-1} p_v^{n-1} - a^2 p_u^k p_v^k - 2c(vp_u^{n-1} + up_v^{n-1})) + \frac{b}{4uv} + 1. \tag{4.18}$$

At $c = 0$ we have discussed this system before (4.13). Now, we consider the second limiting case by

$$a = c \qquad k = n - 1$$

that simplifies the potential part of the Hamiltonian (4.18).

As above, the point canonical transformation (4.14) converts the Hamiltonian (4.18) into the following form:

$$H = p_x^k + p_y^k + \alpha(xy)^{-\frac{k}{k+1}} + \beta$$

after multiplying on a suitable constant and rescaling parameters.

Thus, we present a family of two-dimensional integrable systems, which includes the Kepler and Fokas–Lagerstrom potentials simultaneously.

5. Conclusion

This paper continues the previous works [2, 14, 19, 20]. We propose two new examples of canonical transformations of the extended phase space, which preserve integrability. As a starting point for our consideration we used the known Maupertuis–Jacobi mapping and the Kepler change of the time.

All the known examples and these new examples have many common properties. By studying these properties we hope to obtain a regular method for a search for new integrable systems. On the other hand it allows us to join different integrable systems into the classes of topologically equivalent systems.

The other aim of this paper is to show some new integrable systems related to the Toda lattices and the Stäckel integrable systems.

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